GAUSS EQUATION AND INJECTIVITY RADII FOR SUBSPACES IN SPACES OF CURVATURE BOUNDED ABOVE

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Abstract. A Gauss Equation is proved for subspaces of Alexandrov spaces of curvature bounded above by K. That is, a subspace of extrinsic curvature $\leq A$, defined by a cubic inequality on the difference of arc and chord, has intrinsic curvature $\leq K + A^2$. Sharp bounds on injectivity radii of subspaces, new even in the Riemannian case, are derived.

1. Introduction

Alexandrov spaces are metric spaces with curvature bounds in the sense of local triangle comparisons with constant curvature spaces. In this paper, we consider spaces of curvature bounded above (CBA), and their global counterparts, CAT(K) spaces. Examples include Riemannian manifolds with upper sectional curvature bounds, possibly with boundary ([ABB1]), polyhedra with link conditions, and Tits boundaries and asymptotic cones (see [BH]). A key property of CAT(K) spaces is their preservation under Gromov-Hausdorff convergence. CAT(K) spaces are appropriate target spaces in harmonic map theory (see, for example, [EF, GS, J]), and play an important role in geometric group theory (see [BH]).

Analogues of the Gauss Equation, governing the passage of curvature bounds to subspaces from ambient spaces, tend to be challenging in Alexandrov spaces. For instance, a major unsolved problem in the theory of spaces of curvature bounded below is whether the boundary of a convex set inherits the curvature bound. Somewhat more is known for CBA. A classical theorem of Alexandrov states that a curvature bound above is inherited by ruled surfaces [A]. It is an open problem whether saddle surfaces inherit such a bound, but Mese showed that minimal surface immersions do so [M], and Petrunin [P], that "metric minimizing" surfaces do so.

Recently Lytchak proved that if M has curvature bounded above by K, and N is a complete subset for which there exists $\rho > 0$ such that intrinsic distances $d_N = s$ and extrinsic distances $d_M = r$ satisfy $s - r \leq Cr^3$ for $r < \rho$ (i.e., N is *(C,2,ρ)-convex*), then N has *some* intrinsic curvature bound above [L1]. For a subset N of a Riemannian manifold, an equivalent condition on N is *positive reach*, namely, uniform neighborhoods in which N has the unique footpoint property [L2]. In general, let us say N *is a subspace of extrinsic curvature* $\leq A$ in M if there is a length-preserving map $F : N \to M$ between intrinsic metric spaces, where N is

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complete and

$$s - r \leq \frac{A^2}{24} r^3 + o(r^3) \tag{1}$$

for all pairs of points having s sufficiently small. (For Riemannian submanifolds, this is equivalent to a bound, $|II| \leq A$, on the second fundamental form.) Then points of N have $(C, 2, \rho)$-convex neighborhoods. It follows from [AB3] that points of N have neighborhoods in which r is at least the chordlength of an arc of constant curvature A and length s in the model plane S_K (see Theorem 6.1 below).

In this paper, we extend the Gauss Equation to Alexandrov spaces of curvature bounded above, by proving the following sharp bound for subspaces of extrinsic curvature $\leq A$.

Theorem 1 (Gauss Equation)**.** *Suppose N is a subspace of extrinsic curvature $\leq A$ in an Alexandrov space of curvature bounded above by K. Then N is an Alexandrov space of curvature bounded above by $K + A^2$.*

Remark 2. This bound is realized by constantly curved hypersurfaces of Euclidean, spherical and hyperbolic spaces. At first one might think that Riemannian submanifolds of higher codimension offer a counterexample to this theorem, and that the correct bound should be $K + 2A^2$. On closer inspection, however, one sees that for any plane section, normals to the submanifold may be chosen so that at most two of the corresponding subdeterminants of II are nonzero and one of them is nonpositive. Therefore for Riemannian submanifolds, while the sharp *lower* bound is $K - 2A^2$ when ambient curvature is $\geq K$, the sharp *upper* bound is $K + A^2$ when ambient curvature is $\leq K$.

There are important classes of subspaces for which we can compute sharp extrinsic curvature bounds, and hence sharp intrinsic curvature bounds by Theorem 1. Fibers of warped products are such a class. Warped products of Alexandrov spaces extend standard cone and suspension constructions from one-dimensional to arbitrary base, and gluing constructions from 0-dimensional to arbitrary fiber [AB1]; we expect them to be a major source of constructions and counter-examples in the Alexandrov setting. Theorem 1 allows us to calculate the intrinsic curvature bound of the fiber of a CAT(K) warped product, as we shall discuss in detail elsewhere.

Another significant application of Theorem 1 is to injectivity radii. Theorem 3 gives a sharp estimate on the injectivity radius of a subspace of bounded extrinsic curvature, in terms of the circumference $c(A, K)$ of a circle of curvature A in the simply connected, 2-dimensional space form S_K of curvature K. As always, we set $\pi/k = \infty$ if $k \leq 0$.

Theorem 3. *Suppose N is a subspace of extrinsic curvature $\leq A$ in a CAT(K) space. Then*

$$\mathrm{inj}_N \geq \min\{\frac{\pi}{\sqrt{K + A^2}}, \frac{1}{2} c(A, K)\}. \tag{2}$$

The definition of the injectivity radius inj_N agrees with the usual one when N is Riemannian or locally compact. In the non-locally compact case, a slightly stronger definition is more appropriate (see §5). Even in the case of Riemannian manifolds, our estimates on the injectivity radius of a submanifold are new as far as we know. Much weaker dimension-dependent estimates have been used in [Co] and [S]. The

existence of some dimension-independent bound in the general case is proved in [L2].

The following corollary holds, in particular, for Riemannian submanifolds with $|II| \leq A$ in a Hadamard manifold. Part 2 is an "immersion implies embedding" theorem, which in the case of Riemannian hypersurfaces appeared in [Ar].

Corollary 4. *Let N be a subspace of extrinsic curvature $\leq A$ in a CAT(0) space M.*

1. *N has injectivity radius at least π/A, and any closed ball of radius $\pi/2A$ in N is CAT(A^2).*
2. *If M is CAT($-A^2$), then N is CAT(0) and embedded.*

An interesting example to which Theorems 1 and 3 apply is that of tubular neighborhoods of convex sets. Their extrinsic curvature is analyzed in §6 below. A subset is *π-totally-convex* if it contains every geodesic of length $< \pi$ joining pairs of its points.

Example 5. *Let T be a π-totally-convex set in a CAT(1) space M, and N be the set of points at distance $\leq \rho$ from T, where $\rho < \pi/2$. Then N has CBA by* $\sec \rho$*, and injectivity radius $\geq \pi \cos \rho$.*

2. Outline of paper

A Riemannian manifold has curvature bounded above by K if and only if the lengths of normal Jacobi fields satisfy $f'' \geq -Kf$ in the barrier sense. (In this paper, a continuous function with this property will be called, briefly, *K-convex*.) The same formulation was extended to Riemannian manifolds with boundary in [ABB1], and guides us now in the proof of Theorem 1. However, there we made strong use of the smooth Gauss Equation, which is one of the cornerstones of Riemannian geometry. In contrast, our task here is to find a new approach to that theorem which holds in a far more general setting.

To estimate the curvature of the subset N, we may start with the knowledge that N has some upper curvature bound, and with Lytchak's idea of bounding the extrinsic curvature of two nearby N-geodesics in their ambient ruled surface by projecting to N [L1]. Obtaining a sharp bound for this curvature (Step 2 below) is the difficult core of our paper.

A *geodesic* is an isometric embedding of an interval unless otherwise described. Here are the first two steps of the proof of Theorem 1:

Step 1. For a fan of geodesics γ in N converging to a base geodesic σ, the geodesic chords in M connecting $\gamma(t)$ to $\sigma(t)$ become arbitrarily close to normal to σ and γ, forming a ruled surface R_γ with curvature bounded above by K.

Step 2. The extrinsic curvature of σ and γ in R_γ is bounded above by $A^2(1+\epsilon)d(\sigma(t), \gamma(t))/2$, where $\epsilon \to 0$ as $\gamma \to \sigma$.

Let us describe the geometry behind Step 2. By definition, we must estimate the difference in lengths between a subarc σ_1 of the N-geodesic σ and its chord ν in R_γ, where the *chord* of an arc is the geodesic between its endpoints. Lytchak's bound is based on the estimate

$$\ell(\sigma_1) - \ell(\nu) \leq \ell(\pi_N \nu) - \ell(\nu), \tag{3}$$

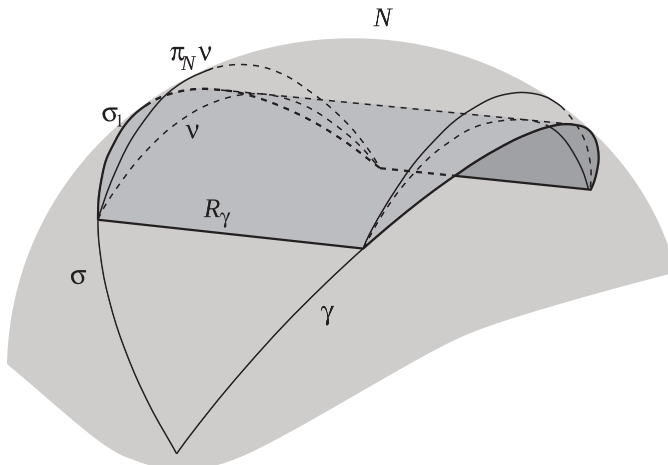

FIGURE 1. Geometric layout

where π_N is projection to N. See Figure 1. Equation (3) holds because the N-geodesic σ_1 is no longer than the curve $\pi_N \nu$ with the same endpoints.

In seeking a sharp bound, we exploit estimates from a globalization theorem for arc/chord curvature in [AB3] (Theorem 6 below). In consequence (Lemma 11 below), we obtain a bound for $\ell(\pi_N \nu) - \ell(\nu)$ which in combination with Equation (3) yields an upper curvature bound for N of $K + 2A^2$. In other words, it turns out that Equation (3) is too weak to give the Gauss Equation we seek; a factor of 2 is needed on the lefthand side. Figure 1 suggests where to look for this factor of 2. Namely, as $\ell(\sigma_1) \to 0$, it seems that $\pi_N \nu$ may be as much longer than σ_1, as σ_1 is longer than ν! Indeed, this is what we prove.

We use a version for CAT(K) spaces (Lemma 8 below) of a standard lemma in smooth comparison geometry. The latter is a length comparison between two curves with the same expression in Fermi coordinates, in ruled surfaces generated by vector fields that are parallel along and orthogonal to two base geodesics (see [CE, "Corollary of Rauch II", p. 31-32]). For the longer of the two curves, the parallel condition can be dropped, since allowing the ruled surface generators to "twist" as they move along the base geodesic only lengthens a curve with given Fermi coordinates.

The remaining steps in the proof of Theorem 1 are:

Step 3. The distance between the respective chords in R_γ of two subarcs of σ and γ is $(K + \epsilon)$-convex. Adding to this the width estimates that follow from Step 2, we find that $d(\sigma(t), \gamma(t))$ satisfies the $(K + \epsilon + A^2(1+\epsilon))$-convexity midpoint inequality.

Step 4. The usual behavior of normal and tangential Jacobi field lengths extends to CBA spaces. Normal Jacobi field lengths along σ are $(K + A^2)$-convex and so, by a development argument, N has CBA by $K + A^2$.

In [ABB1, ABB2], Jacobi fields were studied in Riemannian manifolds with boundary, which form a class of CBA spaces by Alexander et al. [ABB1]. In

Step 4, we must capture the notion of normal and tangential Jacobi fields lengths in the general CBA setting. This is because tangential ones, which are linear, must be discarded in order to identify negative curvature bounds.

Throughout, we may apply the following globalization theorem to N-geodesics, which have extrinsic curvature $\leq A$ in M because N does. The *base angles* of an arc γ are the angles it makes with its chord at their common endpoints, and the *width* of γ is the smallest tubular radius containing γ about the chord. A *k-curve* in S_K is a curve of constant extrinsic curvature k; thus a complete k-curve is a circle, geodesic line, horocycle or equidistant curve.

Theorem 6 (AB3). *Let γ be a curve of extrinsic curvature $\leq k$ in a CAT(K) space.*

1. *If the sum of the arclength and chordlength of γ is $< 2\pi/\sqrt{K}$, then γ has the same arclength and chordlength as a k'-curve in S_K, for some $k' \leq k$.*
2. *If γ is closed (not necessarily closing smoothly) and nonconstant, then γ is no shorter than a complete k-curve (necessarily a circle) in S_K.*
3. *If γ has length $\leq$ half a complete k-curve in S_K, then the base angles and width of γ are no more than they are for a k-curve in S_K of the same length.*

Remark 7. The definition of *extrinsic curvature* $\leq K$ used to prove the theorem above was slightly weaker than that used in this paper. Here, inequality (1) is imposed uniformly at all points of N, whereas above it was imposed on the distances from each point separately. We may take the weaker form in this paper if N is a geodesic space (e.g., if N is locally compact), since then uniformity follows from Part 1 above.

3. Majorizing Lemmas

By a *CAT(K)* space, we mean here a complete metric space in which any two points at distance $< \pi/\sqrt{K}$ are joined by a geodesic, and for any geodesic triangle $\triangle$ of perimeter $< 2\pi/\sqrt{K}$, the distances between points of $\triangle$ are $\leq$ the distances between corresponding points on the triangle with the same sidelengths in S_K. A space has *curvature bounded above (CBA) by* K if every point has a CAT(K) convex neighborhood. As references see [BH, BBI].

An important tool for studying curves in CBA spaces is Reshetnyak's Majorization Theorem [R]:

RMT. *Let γ be a closed curve of length $< 2\pi/\sqrt{K}$ in a CAT(K) space M. Then there is a closed curve $\widetilde{\gamma}$ which is the boundary of a convex region D in S_K and a distance-nonincreasing map $\varphi : D \to M$ such that the restriction of φ to $\widetilde{\gamma}$ is an arclength-preserving map onto γ.*

Such a map φ is called a majorizing map for γ. Note that it is an immediate consequence of the minimizing property of geodesics that for a geodesic subarc of γ, the corresponding subarc of $\widetilde{\gamma}$ is also a geodesic segment; hence, RMT is a broad generalization of the defining property of a CAT(K) space, as is seen by taking γ to be a triangle. Moreover, an extrinsic curvature bound at a point of γ is inherited by the corresponding point of $\widetilde{\gamma}$.

Our first application of the RMT is to prove the Fermi lemma on curvelengths mentioned in the preceding section. The setting is a CAT(K) space M where the nearest-point projection π_ν to a given nontrivial geodesic segment ν is well-defined and continuous. (This will always hold when $K \le 0$, and holds when $K > 0$ under standard size restrictions; see [BH, p. 176-178].) Choose a geodesic $\widetilde{\nu}$ in S_K, where ν and $\widetilde{\nu}$ are parametrized by arc length u on the same interval $[0,r]$, $r > 0$. Define $\psi : M \to S_K$ as follows: $\psi(M)$ lies on one side of $\widetilde{\nu}$; if $\pi_\nu(p) = \nu(u)$, then $d(p,\nu(u)) = d(\psi(p),\widetilde{\nu}(u))$, and the geodesic segment from $\psi(p)$ to $\widetilde{\nu}(u)$ is normal to $\widetilde{\nu}$.

Lemma 8 (Fermi Lemma)**.** *The map $\psi : M \to S_K$ is distance-nonincreasing. If $d(\psi(p),\psi(q)) = d(p,q)$, then $p, q, \pi_\nu(q), \pi_\nu(p)$ are the vertices of a quadrilateral region isometric to a region in S_K.*

Proof. For $p, q \in M$, let p', q' be their projections to ν. If $p' = q'$, then $\psi(p) = \psi(q)$. If $p' \ne q'$, consider the quadrilateral $pqq'p'$. The angles at p', q' are nonacute, so that if we take a majorizing quadrilateral $\tilde{p}\tilde{q}\tilde{q}'\tilde{p}'$ with base $\tilde{p}'\tilde{q}'$ at the corresponding points of $\widetilde{\nu}$, in accordance with RMT, then the base angles remain nonacute. Now we deform $\tilde{p}\tilde{q}\tilde{q}'\tilde{p}'$ so that it has the same base segment and its sides are normal to $\widetilde{\nu}$. The new top vertices are $\psi(p), \psi(q)$. The top side is seen to be shorter than $d(p,q)$ because the deformations of $\tilde{p}$ and $\tilde{q}$, along the circles centered at $\tilde{p}'$ and $\tilde{q}'$ that determine the new quadrilateral, steadily decrease the length of the top side. $\square$

We also require the following local estimates on a subset N of bounded extrinsic curvature. By [L2], in a sufficiently small ambient neighborhood of a point of N, the nearest-point projection π_N is uniquely defined and continuous. Part 2 below sharpens Lytchak's estimate of $1 + Cd(x,N)$ for a Lipschitz constant for π_N on a neighborhood of a point x [L1].

Lemma 9. *Let N be a subset of extrinsic curvature $\le k$ in a CAT(K) space M, and ν be a curve in M parametrized by arclength $u \in [0,r]$.*

1. *If ν is a geodesic in M with ends on N, then*
$$d(\nu(u), N) \le \frac{k}{2}u(r-u) + O(r^4).$$
2. *If ν is any curve within the injectivity distance to N, then*

$$\ell(\pi_N \nu) \le r + \int_0^r [kd(\nu(u),N) + O(d(\nu(u),N)^2)]du. \tag{4}$$

Proof. 1. Recall that an N-geodesic has extrinsic curvature $\le k$ in M. Let σ be an N-geodesic connecting the ends of ν. Consider the closed curve formed by σ and its chord ν. By RMT, this curve is majorized by a map from a region in S_K bounded by a convex arc of curvature $\le k$ and its chord τ of length r. Parametrizing τ by arclength $u \in [0,r]$, we know that the distance from $\tau(u)$ to the convex arc is at most equal to the distance from $\tau(u)$ to the k-curve with the same chord τ and lying on the same side of τ (for details, see [AB3]). Setting $v = r - u$, the latter distance may be calculated as

$$w(u) = \frac{k}{2}uv\left(1 + \frac{k^2}{64}uv + \frac{K}{12}(u^2 + 3uv + v^2) + O(r^4)\right).$$

(When $u = \frac{r}{2}$ this reduces to the expression for the width of the closed curve in terms of r; the corresponding formula for width in terms of the length of the k-curve was given in [AB3].) Thus

$$d(\nu(u), N) \leq d(\nu(u), \sigma) \leq w(u) = \frac{k}{2}uv + O(r^4).$$

$\square$

Proof. 2. Set $d(u) = d(\nu(u), N)$. We are going to construct a curve in S_K of length $r = \ell(\nu)$, parametrized by arclength $u \in [0, r]$, which projects with the *same* distance function $d(u)$ into a complete k-curve. Moreover, if $\widetilde{\sigma}$ is the image arc of this projection, then

$$\ell(\pi_N \nu) \leq \ell(\widetilde{\sigma}). \tag{5}$$

In consequence, we need only verify Equation (4) when $M = S_K$, $N = \widetilde{\sigma}$, and ν is any curve of length r. In this case, the Lipschitz constants of the tangent maps of the projection to $\widetilde{\sigma}$ may be calculated from the lengths of normal Jacobi fields in S_K along geodesics radiating orthogonally to $\tilde{\sigma}$. For example, if $K > 0$, we can take the length of the Jacobi field to be $\sin(\sqrt{K}t)$; the length at points of $\tilde{\sigma}$ is given by taking $t = t_1 < \pi/2$, where $\cot(\sqrt{K}t_1) = k$, while the length at distance d from $\tilde{\sigma}$ is given by taking $t = t_1 - d$. This gives a Lipschitz constant of

$$\frac{\sin(\sqrt{K}t_1)}{\sin(\sqrt{K}[t_1 - d])} = 1 + kd + (k^2 + \frac{K}{2})d^2 + O(d^3).$$

The series expression for the ratio of Jacobi field lengths in terms of d and K persists for all values of K. The bound (4) on $\ell(\pi_N \nu)$ now follows by integrating this ratio.

To carry out the construction, partition ν into subarcs, and apply RMT to each closed figure whose base is a chord of that subarc, whose sides are the minimizers from the endpoints of the base to N, and whose top is the N-geodesic joining the footpoints of these minimizers. The corresponding convex curve in S_K has a geodesic base, two geodesic sides, and a top curve of curvature $\leq k$ making nonacute angles with both sides; all of these arcs have the same length as their corresponding arcs in M. Now replace the top curve in S_K by the k-curve with the same endpoints; this move does not decrease the length of the top or the angles between the top and sides. The latter angles are nonacute and, if not right angles, may now be made right by hinging the sides inward, thereby reducing the length of the base. The final move on each figure in S_K is to lengthen the top k-curve, preserving its right angles with the sides and the lengths of the sides, until the baselength is restored to that of the corresponding segment of ν. For any partition of ν, we now glue the resulting figures in S_K along corresponding sides. By construction, the top curves form a k-curve whose length is at least that of a broken N-geodesic approximation of $\pi_N(\nu)$. Thus we have constructed a polygonal curve in S_K which converges, as the original partition of ν is refined, to a curve of the same length as ν. By construction, this curve projects, with the same distance function as $\pi_N|\nu$, to a k-curve $\widetilde{\sigma}$ satisfying (5). $\square$

The following lemma gives width and base angle estimates that are a direct consequence of Theorem 6.3 and the power series expansions in [AB3, Remark 6.2]:

Lemma 10. *For a curve of extrinsic curvature $\leq k$ in a CAT(K) space, the base angles φ and width W of an arc with chordlength r satisfy*

1. $\varphi \le kr/2 + O(r^3)$,
2. $W \le kr^2/8 + O(r^3)$.

4. Proof of Theorem 1 (Gauss Equation)

Now we are ready to prove our Gauss Equation, namely, a sharp curvature bound for a subset N of extrinsic curvature $\le A$ in an Alexandrov space M of CBA by K. The proof breaks into Steps 1 to 4, as outlined in §2.

4.1. **Step 1: The ruled surfaces R_γ.** By a *fan* in N, we mean a one-parameter family of N-geodesics $\gamma = \gamma_u$ in N, originating at a point p and with righthand endpoints moving along a geodesic with parameter u. (Recall that all geodesics are parametrized by arclength.) Set $\gamma_0 = \sigma$. Since N has some upper curvature bound $L > 0$, then the balls centered at p are convex as long as they are contained in a CAT(L) neighorhood of p and the distance from p is $< \pi/2\sqrt{L}$; we assume that the generators γ_u of the fan are shorter than that radius.

The function $f(t) = d(\sigma(t), \gamma(t))$ is L-convex and increasing. Here, L-convexity follows from RMT and the fact that the distance between geodesics in the model space S_L is L-convex for $L \ge 0$. By extracting a subsequence γ_i of the fan generators, and setting $f_i = d(\sigma, \gamma_i)$, we may assume the existence of

$$F(t) = \lim_{i\to\infty} u_i^{-1} f_i(t). \tag{6}$$

Moreover, F is continuous and L-convex on $[0, \ell_0)$, where $\ell_0 = |\sigma|$. (See the argument in [ABB2, p. 178].) The function F plays the role of a normal Jacobi field *length*; we do not need the notion of Jacobi field *direction*.

Recall that L-convex functions have one-sided derivatives, and derivatives almost everywhere. Since, moreover, L-convex functions converge with their one-sided derivatives, it follows from Equation (6) that the one-sided derivatives f_i' are $\le Cu_i$, hence converge unformly to 0. By first variation, $f_i'(t_+) = -\cos\alpha_i(t) - \cos\beta_i(t)$, where $\alpha_i(t) = \angle(\sigma(t)\gamma_i(t)\gamma_i(\ell_0))$ and $\beta_i(t) = \angle(\gamma_i(t)\sigma(t)\sigma(\ell_0))$. By convexity of balls, $\alpha \ge \pi/2, \beta \ge \pi/2$. Hence α_i, β_i are uniformly close to $\pi/2$ as $i \to \infty$. (If f_i vanishes on an initial interval, we set $\alpha_i = \beta_i = \pi/2$ there.) Similarly, the "supplementary" angles $\angle(\sigma(t)\gamma_i(t)p)$ and $\angle(\gamma_i(t)\sigma(t)p)$ are also uniformly close to $\pi/2$. This shows that the ruled surface in N, formed by connecting the pairs of points $\gamma_i(t)$ and $\sigma(t)$ by N-geodesics, has its rulings nearly normal to γ_i and σ.

When we form the corresponding ruled surface R_{γ_i} in M, the rulings are chords of the N-geodesic rulings, where the latter have extrinsic curvature $\le A$ by hypothesis. Theorem 6.3 gives an upper bound for the angles between the two kinds of rulings, for i sufficiently large (depending only on A, K). Therefore the rulings of R_{γ_i} may be assumed arbitrarily close to normal to γ_i and σ.

By Alexandrov's theorem on ruled surfaces, R_{γ_i} has curvature bounded above by K.

4.2. **Step 2: Extrinsic curvature of σ & γ in R_γ.** In this subsection, we use γ to denote the N-geodesic γ_i for i sufficiently large. We shall prove that the extrinsic curvature of σ and γ in the ruled surface R_γ is bounded above by $A^2(1+\epsilon)f(t))/2$, where $f = d(\sigma, \gamma)$ and $\epsilon \to 0$ as γ approaches σ, that is, as $i \to \infty$.

We are going to develop two ways of estimating the length of the projection to N of a chord ν of σ in R_γ: an upper bound for $\ell(\pi_N(\nu))$, by involving the curvature of σ in R_γ to bound the distances $d(\nu(u), N)$; and a lower bound, by involving that

curvature to estimate the distances of $\pi_N(\nu)$ from its chord σ in N (see Figure 1). The combination of these two bounds will yield the desired inequality on the curvature of σ. The argument is symmetric, giving the same bound for γ.

Let k denote the least curvature bound of σ in R_γ, i. e., for every subarc of σ of length s and chord length r in R_γ we have $s - r \leq \frac{k^2}{24}s^3 + o(s^3)$, and k is the least such number.

By Step 1, we may assume that the rulings η_t of R_γ make angles with σ and γ which differ from $\pi/2$ by at most δ. Consider a subarc of σ with chord ν of length r in R_γ. See Figures 1 and 2. A triangle $\triangle\nu(u)mq$ in R_γ formed by a point $\nu(u)$ on the chord, its nearest point m on σ, and the point $q = \sigma(t)$ where the ruling η_t through $\nu(u)$ ends on σ, has at least one side of magnitude $d(\nu(u), m) = O(r^2)$, by

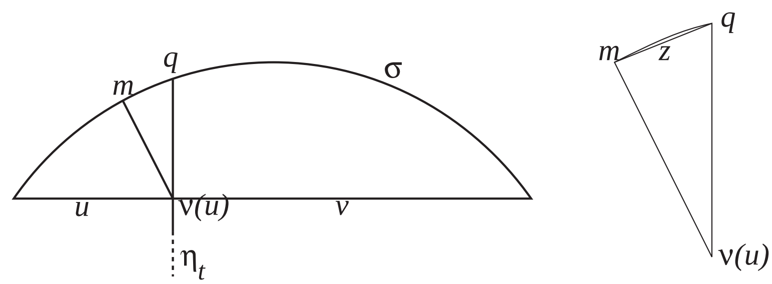

FIGURE 2. Arcs and chords in R_γ

Lemma 10.2. The other two sides are surely at most $O(r)$; hence the model area is $O(r^3)$, and the angle sum is bounded by $\pi + O(r^3)$.

First we get an upper bound on the length of the projection to N of ν:

Lemma 11. *For a chord ν of σ, of chordlength r:*

$$\ell(\pi_N\nu) \leq r + \frac{r^3}{24}A^2k(1 + \delta^2/2 + O(r^2))f(t)(1 + O(f(t)^2)), \tag{7}$$

where $f(t) = d(\sigma(t), \gamma(t))$.

Proof. By Lemma 9.1,

$$d(\nu(u), m) \leq \frac{k}{2}uv + O(r^4). \tag{8}$$

(Distances between points of R_γ are the distances within R_γ.) Let $\alpha = \angle\nu(u)mq$, $\beta = \angle\nu(u)qm$, $\theta = \angle m\nu(u)q$, and $z = d(m, q)$. The base angles formed by the chord mq and the subarc of σ with ends m, q are bounded by $\frac{k}{2}z + O(z^3)$, by Lemma 10.1, so that we get lower bounds on α, β:

$$\alpha - [\frac{\pi}{2} - \frac{k}{2}z - O(z^3)] \geq 0,\ \beta - [\frac{\pi}{2} - \delta - \frac{k}{2}z - O(z^3)] \geq 0. \tag{9}$$

The angles of the model triangle of $\triangle\nu(u)mq$ in S_K satisfy $\widetilde{\alpha} \geq \alpha$, $\widetilde{\beta} \geq \beta$, $\widetilde{\theta} \geq \theta$. Add the inequalities (9) with α, β replaced by $\widetilde{\alpha}, \widetilde{\beta}$ to the obvious inequality $\widetilde{\theta} \geq 0$ and $\widetilde{\alpha} + \widetilde{\beta} + \widetilde{\theta} \leq \pi + O(r^3)$; then the summands π and $-\frac{\pi}{2} - \frac{\pi}{2}$ cancel, leaving

$\delta + k + O(z^3) + O(r^3)$ as an upper bound for the sum of three nonnegative terms. Hence each term must have that same upper bound, which gives the following.

$$
\begin{aligned}
\widetilde{\alpha} &\le \frac{\pi}{2} + \delta + \frac{k}{2}z + O(z^3) + O(r^3), \\
\widetilde{\beta} &\le \frac{\pi}{2} + \frac{k}{2}z + O(z^3) + O(r^3), \\
\widetilde{\theta} &\le \delta + kz + O(z^3) + O(r^3).
\end{aligned} \tag{10}
$$

We shall apply the law of sines to $\triangle \widetilde{\nu}(u)\widetilde{m}\widetilde{q}$ to estimate z and $d(\nu(u), q) = d(\widetilde{\nu}(u), \widetilde{q})$. In doing so we may use the Euclidean version, since the higher order terms in $\sin(\sqrt{K}z)$ and $\sin(\sqrt{K}d(\nu(u), q))$ can be absorbed in the error term $O(r^3)$. Thus we have, from (8),

$$
z = \frac{\sin\widetilde{\theta}}{\sin\widetilde{\beta}} d(\nu(u), m) + O(r^3) \le \frac{\delta + kz}{\sin\widetilde{\beta}} kuv + O(r^3).
$$

To continue we require a lower bound on $\sin\widetilde{\beta}$. Note that the range of $\widetilde{\beta}$ is bounded above by (10), and below by $\widetilde{\beta} \ge \beta \ge \pi/2 - (\delta + \frac{k}{2}z)$ from (9). A lower bound on $\sin\widetilde{\beta}$ is obtained by checking the endpoints. On the right, since $z = O(r^2)$, we have $\sin[\frac{\pi}{2} + \frac{k}{2}z + O(z^3) + O(r^3)] = 1 + O(r^4)$. Hence, we must use the left, namely, $\sin\widetilde{\beta} \ge \cos(\delta + \frac{k}{2}z) = 1 - \frac{\delta^2}{2} + O(r^2)$ and $(\sin\widetilde{\beta})^{-1} \le 1 + \frac{\delta^2}{2} + O(r^2)$. This gives:

$$
z \le kuv(\delta + \frac{\delta^3}{2}) + O(r^4). \tag{11}
$$

In turn the law of sines also gives $d(\nu(u), q) = (\sin\widetilde{\alpha}/\sin\widetilde{\beta}) d(\nu(u), m)$, and so:

$$
d(\nu(u), q) \le \frac{1}{1 - \delta^2/2} \frac{kuv}{2} + O(r^4) = \frac{k(1 + \delta^2/2)}{2} uv + O(r^4). \tag{12}
$$

Since the geodesic in N having η_t as chord has curvature $\le A$ in M, we can also apply Lemma 9.1 to estimate $d(\nu(u), N)$. Accordingly, in Lemma 9.1, replace u, r, v by $\bar{u} = d(m, q)$, $\bar{r} = f(t) = d(\sigma(t), \gamma(t))$, $\bar{v} = \bar{r} - \bar{u} \le f(t)$, obtaining

$$
\begin{aligned}
d(\nu(u), N) &\le \frac{A}{2} d(\nu(u), q) f(t)(1 + O(f(t)^2)) \\
&\le \frac{Ak(1 + \delta^2/2 + O(r^2))}{4} uvf(t)(1 + O(f(t)^2)).
\end{aligned} \tag{13}
$$

Then in Lemma 9.2 we find that the original chord of length r has projection to N of length

$$
\begin{aligned}
\ell(\pi_N\nu) &\le r + \int_0^r u(r-u)du \cdot \frac{A^2k(1 + \delta^2/2 + O(r^2))}{4} f(t)(1 + O(f(t)^2)) \\
&= r + \frac{r^3}{24} A^2k(1 + \delta^2/2 + O(r^2)) f(t)(1 + O(f(t)^2)).
\end{aligned}
$$

□

Now we develop a lower bound for $\ell(\pi_N\nu)$ when ν is a chord which gives a sufficiently good approximation for the curvature bound k. For any given positive number λ there is a subarc σ_1 of length s_1 with chord length r_1 such that $s_1 - r_1 \ge \frac{(k-\lambda)^2}{24} s_1^3$, where $s_1 \to 0$ as $\lambda \to 0$. Below we assume $\lambda < k/6$. The length r_1 can be attained from σ_1 by a process involving first variation: parametrize σ_1 by arclength $s \in [0, s_1]$ and let the base angle of the chord from $\sigma(0)$ to $\sigma(s)$ at $\sigma(s)$ be

$\varphi(s) = \bar{\varphi}(s)s$. Then the first variation formula gives r_1 by integrating the derivative of the length of those chords, i.e.,

$$r_1 = \int_0^{s_1} \cos\varphi(s)\, ds \geq s_1 - \sup \frac{\bar{\varphi}(s)^2}{2} \cdot \frac{s_1^3}{3} + O(s_1^5).$$

Then a point s_2 which approximates $\sup \bar{\varphi}(s)$ sufficiently closely gives us a chord with a relatively large base angle:

$$\bar{\varphi}(s_2)^2 \cdot \frac{s_1^3}{6} \geq \frac{(k-2\lambda)^2}{24} \cdot s_1^3.$$

In this the discrepancy between $\sup \frac{\bar{\varphi}(s)^2}{2} \cdot \frac{s_1^3}{3} + O(s_1^5)$ and an upper bound on $s_1 - r_1$ has been absorbed by changing λ to 2λ. Then

$$\varphi(s_2) \geq \frac{k-2\lambda}{2} s_2. \tag{14}$$

Now we turn our attention to the arc σ_2 of length s_2, parametrized by $s \in [0, s_2]$, with chord ν of length r_2, and base angle $\varphi = \varphi(s_2)$ satisfying (14). For an initial subarc of σ_2 of length s and base angle $\varphi_1(s)$, we use the lower bound (14) on φ and the upper bound $\varphi_1(s) \leq \frac{k}{2}s + O(s^3)$ of Lemma 10.1, to get a lower bound on the distance w from $\sigma_2(s)$ to ν. See Figure 3. The chord of that subarc is the

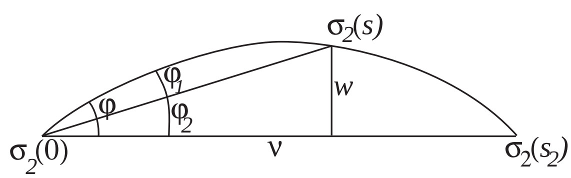

FIGURE 3. Lower bound for w

hypotenuse (with length $\mu \geq s - \frac{k^2}{24}s^3 + O(s^5)$) of a right triangle with angle $\varphi_2(s)$ at $\sigma_2(0)$ and opposite leg of length w. From the triangle inequality for angles we have $\varphi_2(s) \geq \frac{k-2\lambda}{2}s_2 - \frac{k}{2}s + O(s^3)$, so that by side-angle-side comparison we have

$$w \geq \mu \cdot \sin\varphi_2(s) \geq \frac{1}{2}\left(ks(s_2 - s) - 3\lambda s s_2\right). \tag{15}$$

The foot of the leg of length w in the above right triangle projects to N to give a point at distance w' from $\sigma_2(s)$. The triangle inequality gives a satisfactory lower bound for w', using the same bounds on distances to N as in (13),

$$w'(s) \geq w(1-\epsilon) \geq \frac{1-\epsilon}{2}\left(ks(s_2 - 2) - 3\lambda s s_2\right) = y(s), \tag{16}$$

where $\epsilon \to 0$ as $\gamma \to \sigma$.

We let ν be the chord of σ_2, and apply Lemma 8 to the curve consisting of $\pi_N \nu$ and geodesic segment σ_2 in the space N. We may assume that s_2 is so small that the target of ψ can be considered to be the Euclidean plane, omitting the negligible error terms. If we take the piece of the parabola $y = y(x)$ for $x \in [0, (1 - \sqrt{3\lambda/k})s_2]$ and its tangent line through $(0, s_2)$ (the point of tangency determines the interval for the piece), then we have a concave curve shorter than the length $\psi(\pi_N(\nu))$,

which is in turn shorter than $\pi_N(\nu)$. A lengthy but straightforward power series calculation gives a lower bound, for sufficiently small s_2:

$$\ell(\pi_N\nu) \geq s_2 + (1-\epsilon)^2\frac{(k-5\lambda)^2}{24}s_2^3.$$

Just as we obtained a lower bound for $\ell(\pi_N\nu)$, we can obtain a lower bound for the length s_2 of σ_2 by expressing the distances w in terms of the arclength parameter $r \in [0, r_2]$ of its chord, which yields the same expression except for higher order error terms:

$$w \geq \frac{1}{2}\left(kr(r_2 - r) - 3\lambda r r_2\right).$$

As before, Lemma 8 and an integration to obtain the length of a concave curve in the Euclidean plane gives

$$s_2 \geq r_2 + \frac{(k-5\lambda)^2}{24}s_2^3.$$

where $s_2 \to 0$ as $\lambda \to 0$ and we have converted r_2^3 into s_2^3 at the expense of higher order terms. Now we can add these two inequalities to obtain and cancel the summand s_2 to obtain a lower bound

$$\ell(\pi_N\nu) \geq r_2 + (1-\epsilon)^2\frac{(k-4\lambda)^2}{24}s_2^3 + \frac{(k-5\lambda)^2}{24}s_2^3.$$

Now we chain this with the upper bound inequality (7) with $r = r_2$, with r_2^3 replaced by s_2^3. The summands r_2 can be canceled, and in the resulting equation we can divide by s_2^3 to leave an inequality on k:

$$(1-\epsilon)^2\frac{(k-4\lambda)^2}{24} + \frac{(k-5\lambda)^2}{24} \leq \frac{1}{24}A^2k(1+\delta^2/2+O(r^2))f(t)(1+O(f(t)^2)).$$

In this inequality $4\lambda, 5\lambda$ can be removed by letting $s\lambda \to 0$. Then cancelling a factor $k/24$ gives the required estimate, namely, $k \leq A^2(1+\epsilon')f(t))/2$, where $\epsilon' \to 0$ as γ approaches σ.

4.3. **Step 3: $(K + A^2)$-convexity.** As in the preceding step, γ will denote the N-geodesic γ_i for i sufficiently large. We claim that $f = d(\sigma, \gamma)$ is almost $K + A^2$-convex, so that the corresponding "normal Jacobi field length" F defined by Equation (6) of Step 1 is in fact $(K + A^2)$-convex.

To estimate the convexity of f we concentrate on the neighborhood of a single ruling of R_γ; for convenience, shift the parametrization so that this ruling is η_0 and the neighborhood is the strip between η_{-t} and η_t. The ends of the strip are bounded by subarcs of σ and γ and these subarcs will have geodesic chords in R_γ, which we denote by σ' and γ'. See Figure 4. Since R_γ has CBA by K, the distance between corresponding points of σ' and γ' is $(K+\epsilon')$-convex, where $\epsilon' \to 0$ as $\gamma \to \sigma$. Then the distances between the ends are $f(-t), f(t)$, and with the distance h between the points m', q' where η_0 crosses γ', σ' they must satisfy the approximate $K + \epsilon'$ midpoint convexity inequality

$$f(-t) + f(t) - 2h \geq -(K+\epsilon')ht^2 + O(t^3).$$

Now we want to establish the same sort of inequality for $f(-t), f(0), f(t)$. Since η_0 is partitioned by m', q' into three segments, the only difference is the addition of the two end segments, which we have estimated in Step 2. Letting $u = v = t$ and $m = \gamma(0), q = \sigma(0)$, we have that $d(m', m)$ and $d(q', q)$ are both bounded by

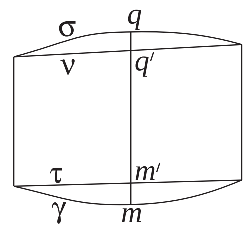

FIGURE 4. Midpoint separations

$A^2(1+\epsilon)f(0)uv/4+O(t^4) = A^2(1+\epsilon)f(0)t^2/4+O(t^4)$. Hence $f(0) = h+d(m',m)+d(q',q) \le h + A^2(1+\epsilon)f(0)t^2/2 + O(t^4)$. Then

$$\begin{aligned} f(-t)+f(t)-2f(0) &\ge f(-t)+f(t)-2h-A^2(1+\epsilon)f(0)t^2+O(t^3) \\ &\ge -[(K+\epsilon')h + A^2(1+\epsilon)f(0)]t^2 + O(t^3) \\ &= -(K+A^2+\epsilon'')f(0)t^2 + O(t^3), \end{aligned}$$

where $\epsilon'' \to 0$ as $\gamma \to \sigma$.

4.4. **Step 4: Jacobi field lengths.** We have been studying a fan of geodesics in N with lefthand endpoints fixed at the vertex of a triangle, and righthand endpoints moving along the opposite side. To conclude that N has CBA by $K + A^2$, we must establish the local triangle comparison property with S_{K+A^2}. The key is our conclusion in Step 3, according to which "normal Jacobi field lengths" F defined as in (6) are $(K + A^2)$-convex. From here, the desired triangle comparison property may be established by developing the fan into S_{K+A^2}, provided we first establish the appropriate splitting property for Jacobi field lengths.

Consider reparametrizations of the γ_i by $[0, \ell_0]$, setting $g_i = d(\gamma_i(\ell_i\ell_0^{-1}t), \sigma(t))$ where $\ell_i = |\gamma_i|$ and $\ell_0 = |\sigma|$. Let $G = \lim_{i\to\infty} u_i^{-1} g_i$. We claim

$$\begin{aligned} G(t)^2 &= F(t)^2 + t^2 \lim_{i\to\infty} u_i^{-2}(\ell_i - \ell_0)^2 \\ &= F(t)^2 + t^2\cos^2\theta, \end{aligned} \tag{17}$$

where $\theta = \angle pqr$.

The second equality of (17) is immediate from the first variation formula. The first equality is equivalent to

$$g_i(t)^2 - f_i(t)^2 - t^2(\ell_i - \ell_0)^2 = o(u_i^2). \tag{18}$$

This expression may be interpreted in terms of the law of cosines for the planar triangle $\overline{\triangle}$ with sidelengths $f_i(t)$, $t(\ell_i - \ell_0)$, and $g_i(t)$, namely, the model triangle for $\triangle = \triangle\sigma(t)\,\gamma_i(t)\,\gamma_i(t\ell_0^{-1}\ell_i)$. Thus the lefthand side of (18) equals

$$f_i(t)t(\ell_i - \ell_0)\cos\overline{\beta}, \tag{19}$$

where $\overline{\beta}$ is the angle of $\overline{\triangle}$ at the vertex corresponding to $\gamma_i(t)$. By triangle comparison, $\overline{\beta} \ge \beta$, where β is the angle of $\triangle$ at $\gamma_i(t)$. By Step 1, β is arbitrarily close to $\pi/2$.

Now apply RMT to the quadrilateral $\square = \square\sigma(t)\,\gamma_i(t)\,\gamma_i(t\ell_0^{-1}\ell_i)\,\sigma(t\ell_0^{-1}\ell_i)$. The corresponding planar quadrilateral $\overline{\square}$ has two adjacent sidelengths agreeing with those of $\overline{\triangle}$, and their diagonal $\ge g_i(t)$, which is the third sidelength of $\overline{\triangle}$. Therefore

$\widetilde{\beta} \geq \overline{\beta}$, where $\widetilde{\beta}$ is the corresponding angle of $\overline{\square}$. Since all four angles of $\overline{\square}$ majorize angles that are, by Step 1, arbitrarily close to $\pi/2$, all angles of $\overline{\square}$, including $\widetilde{\beta}$, must be arbitrarily close to $\pi/2$. Hence so is $\overline{\beta}$, since it lies between $\widetilde{\beta}$ and β. Thus the expression (19), when divided by u_i^2, approaches 0, since $u_i^{-1} f_i(t) \to F(t)$, $u_i^{-1}(\ell_i - \ell_0) \to \cos\varphi$, and $\cos\overline{\beta} \to 0$. This completes the verification of (17).

The triangle comparison property now may be established by development into S_{K+A^2}, as in [ABB1, p. 709-710]. The argument requires that in a neighborhood of each point of N, geodesic variations are Lipschitz if their endpoint curves are. This holds because N is known to have some upper curvature bound. Specifically, if M is CAT(K) then N is CAT(k) for some sufficiently large k [L1]. In this case, we conclude that N has CAT($K + A^2$) neighborhoods of uniform size.

5. Injectivity radius bounds

5.1. **Injectivity radius.** If M is complete and locally compact, the *injectivity radius* $\mathrm{inj}_{M.p}$ is the infimum of sidelengths of *lunes* (pairs of distinct geodesics with common endpoints) from p. For general inner metric spaces, following a suggestion of Lytchak [L5], we use a slightly stronger definition of $\mathrm{inj}_{M,p}$, namely: the supremum of radii ρ with the property that for any $\epsilon > 0$ there exists $\delta > 0$ such that any two Lipschitz curves from p to any $q \in B(p, \rho)$, whose lengths are within δ of $d(p, q)$, are ϵ-close to each other. Here we parametrize the curves proportionally to arclength by $[0, 1]$, and use the uniform distance. If M is complete, it follows that balls of radius $< \mathrm{inj}_{M,p}$ possess *radial uniqueness*, by which we mean that any point q is joined to p by a unique geodesic and this geodesic varies continuously with q.

It is easily verified that $\mathrm{inj}_{M,p}$ is the infimum of sidelengths of lunes from p in the ultraproduct M^ω. Here M^ω is defined as an ultralimit of the sequence $M, M, \ldots$ with fixed basepoint (see [BH], [KL, §2.4], [L4]). M^ω is a complete geodesic space, which contains an isometric copy of M, and is isometric to M if M is complete and locally compact. A sequence of quasi-isometries into a bounded subset of M induces a quasi-isometry into M^ω, and a complete metric space M is CAT(K) if and only if M^ω is CAT(K).

5.2. **Proof of Theorem 3.** A *local geodesic* is a locally distance-realizing curve, parametrized proportionally to arclength by $[0, 1]$. In a complete space of curvature bounded above by K, every local geodesic γ from p of length $< \pi/\sqrt{K}$ has a neighborhood, in the space of local geodesics from p in the uniform topology, on which the righthand endpoint map is injective. The size of this neighborhood is uniformly bounded below in terms of K, the length of γ, and the least size of CAT(K) neighborhoods of points of γ. (See [AB4], where it is proved that this neighborhood may be taken so that the endpoint map is a homeomorphism onto a neighborhood of the endpoint of γ.) We say, M has *no conjugate points before* $\pi/\sqrt{K}$.

Now suppose N is a subspace of extrinsic curvature $\leq A$ in a CAT(K) space M. Since N has CAT($K + A^2$) neighborhoods of uniform size, the same is true of N^ω and $(N^\omega)^\omega$.

Suppose, contrary to Theorem 3, that $\mathrm{inj}_{N,p} < \min\{\pi/\sqrt{K + A^2}, c(A, K)/2\}$. There is a sequence of lunes in N^ω from p with sidelengths approaching $\mathrm{inj}_{N,p}$. Since N^ω has no conjugate points before $\pi/\sqrt{K + A^2}$, the sides are uniformly bounded

apart, and so there is a lune from p of sidelength $\mathrm{inj}_{N,p}$ in $(N^\omega)^\omega$. A diagonalization argument gives the existence of such a lune in N^ω. By Theorem 6.2, the sides of this lune cannot meet at angle π, since that would give a closed curve in M with extrinsic curvature $\leq A$ and length $< c(A,K)$. Since there is no shorter lune from p in N^ω, the sides also cannot meet at angle $< \pi$. Indeed, in that case they could be deformed to a pair of shorter local geodesics by first variation; thus to complete the proof, it is only necessary to verify that a local geodesic γ in N^ω from p whose length is less than both $\mathrm{inj}_{N,p}$ and $\pi/\sqrt{K+A^2}$ is a geodesic. But if the maximal minimizing subsegment of γ from p had endpoint $\gamma(t)$, $0 < t < 1$, there would be geodesics from p to $\gamma(t+\frac{1}{n})$ that were bounded away from this subsegment, again since N^ω has no conjugate points before $\pi/\sqrt{K+A^2}$. Then there would be a lune from p in $(N^\omega)^\omega$ of sidelength $< \mathrm{inj}_{N,p}$, and hence by diagonalization a forbidden lune in N^ω.

5.3. **Proof of Corollary 4.** For Part 1, consider a subspace N of extrinsic curvature $\leq A$ in a CAT(0) space M. Since $\pi/\sqrt{A^2} = c(A,0)/2 = \pi/A$, it follows from Theorems 1 and 3 that N has curvature bounded above by A^2 and injectivity radius at least π/A. Therefore balls of smaller radius possess radial uniqueness. By [AB3, Th. 4.3], a ball of radius $\pi/2A$ in N that possesses radial uniqueness is CAT(A^2), as required.

In Part 2, M is assumed moreover to be CAT($-A^2$). Then N has curvature bounded above by 0 by Theorem 1, and infinite injectivity radius by Theorem 3, and hence is CAT(0). If N were not embedded in M, there would be a nonconstant geodesic in N, and hence a curve of extrinsic curvature $\leq A$ in M, with endpoints mapped to a single point of M. Since a complete A-curve in the hyperbolic space of curvature $-A^2$ has infinite length, Theorem 6.2 would be contradicted.

6. Extrinsic curvature of tubular neighborhoods

Finally we obtain sharp extrinsic curvature bounds for tubular neighborhoods of convex sets.

Given the following proposition, the intrinsic curvature bounds in Example 5 follow immediately from Theorem 1. The injectivity radius bounds follow from Theorem 3, since $\sqrt{1+\tan^2\rho} = \sec\rho$ and $c(\tan\rho, 1) = 2\pi\cos\rho$.

Proposition 12. *Let T be a π-totally-convex set in a CAT(1) space M, and let N be the subset of points at distance $\leq \rho$ from T, where $\rho < \pi/2$. Then N has extrinsic curvature $\leq \tan\rho$.*

Proof. We reduce to the special case where T is a geodesic segment in the model space S_1. That reduction proceeds as follows.

Let ν be a geodesic chord of length $< \pi$ in M joining two points of N. Let the endpoints $\nu(0), \nu(r)$ of ν project to $m, q \in T$, and γ be the geodesic connecting m and q in T, where $\ell(\gamma) < \pi$. (For the existence and contracting property of projection to T, see [BH, p. 176].) By RMT there is a convex quadrilateral $\square = \square\widetilde{m}\widetilde{\nu}(0)\widetilde{\nu}(r)\widetilde{q}$ in S_1 and a distance-nonincreasing map $\varphi : \mathrm{Conv}\,\square \to M$, preserving the lengths of the sides of $\square$. (*Conv* indicates the convex hull.) Let $\widetilde{\gamma}$ be the base of $\square$, connecting $\widetilde{m}, \widetilde{q}$, and let $\widetilde{N}$ be the set of points at distance $\leq \rho$ from $\widetilde{\gamma}$. It is easily seen that $\widetilde{N}$ is bounded by two semi-circular arcs of radius $\rho < \pi/2$ about $\widetilde{m}, \widetilde{q}$ with diameters perpendicular to $\widetilde{\gamma}$, joined by two circular arcs at distance ρ

from the great circle which includes $\widetilde{\gamma}$. Thus these joining arcs have curvature $\cot \rho$ and $\widetilde{N}$ is a subset of extrinsic curvature $\cot \rho$. This extrinsic curvature bound is then inherited by $\widetilde{N} \cap \operatorname{Conv} \Box$. Let $\widetilde{\sigma}$ be the geodesic in $\widetilde{N}$ between $\widetilde{\nu}(0), \widetilde{\nu}(r)$, so $\widetilde{\sigma}$ has length $\leq$ an arc of curvature $\cot \rho$ and chord-length r and its points are at distance $\leq \rho$ from $\widetilde{\gamma}$. Using the distance-nonincreasing property, we have that $\varphi(\widetilde{\sigma})$ has no greater length than $\widetilde{\sigma}$ and its points are at distance $\leq \rho$ from γ, and hence also from T, so $\varphi(\widetilde{\sigma})$ is in N. This provides the requisite bound on $d_N(\nu(0), \nu(r))$ corresponding to an extrinsic curvature bound $\tan \rho$ for N. $\Box$

The extrinsic curvature bound in Theorem 12 is sharp, as is shown by the example of T being a geodesic segment in S_1, including the possibility of an entire great circle. The existence of some curvature bound for a ball of radius $< \pi$ (corresponding to taking T a ball of radius $< \pi/2$) was obtained in [L1].

Remark 13. Let $f = \sin \sqrt{K} d_T$ for some $\pi/\sqrt{K}$-totally-convex set T in a $\mathrm{CAT}(K)$ space, $K > 0$. It is proved in [AB2] that f is $\mathcal{F}K$*-convex*, on the subset where $d_T < \pi/2\sqrt{K}$. That is, the restriction of f to every geodesic is K-convex. $\mathcal{F}K$-convex functions are basic to the study of spaces with curvature bounds (see [AB2, G]). Proposition 12 is a special case of the following statement, which seems likely to be true, but which we shall not pursue here: for an $\mathcal{F}K$-convex function f on a $\mathrm{CAT}(K)$ space, $K > 0$, the sublevel set $f^{-1}(-\infty, c]$ for $c \geq 0$ has extrinsic curvature $\leq Kc/\inf |Df|$, where the infimum is taken on the level set $f^{-1}(c)$. (Since the sublevels for $c \leq 0$ are convex, the statement is certainly true for $c = 0$.) It follows from [L3, Cor 1.9] that there is some extrinsic curvature bound whenever the gradient is bounded from below. For $K < 0$, the corresponding statement would be for nonpositive values of c, or what is the same, for nonnegative superlevel sets of an $\mathcal{F}K$-concave function.

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